A DEVELOPMENT AND COMPARATIVE ANALYSIS OF TWO MODELS FOR AIR DEFENSE GUN SYSTEM BURST KILL PROBABILITY

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max formula results.

for model I

3xP for model II a

3xP for II b

THESIS

A Development and Comparative Analysis
of Two Models for Air Defense Gun
System Burst Kill Probability

by

David Kenneth Heebner

March 1976

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System Burst Kill Probability

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I. INTRODUCTION TO THE PROBLEM

Air defense gun systems have maintained an important role in air defense operations despite the advent of effective surface-to-air missiles. Reference 1 reports that the U. S. Air Force lost 550 aircraft to air defense guns in the Korean Conflict, and more recently, U. S. forces lost more than 3000 fixed and rotary wing aircraft to enemy action in the Vietnam War with just a small percentage of these losses due to surface-to-air missiles. Furthermore, these losses do not reflect the considerable aircraft repair burden imposed by air defense guns nor the contributing effects (short of aircraft destruction) that guns have in deterring piloted aircraft from the accomplishment of their objectives.

The air defense objectives have remained relatively constant over time, but the problems and technology of the gun/target engagement have changed. Helicopters are now an integral part of the battlefield, tactical aircraft are faster and more maneuverable, contour flying is possible under adverse conditions and air-to-surface weapon systems can be incredibly sophisticated. At the same time, digital computers, improved sensor capabilities and surface-to-air missiles which force higher targets to low altitude have favored the efficacy of the air defense gun system.

Considering this continually improving capability of aircraft to deliver their ordinance effectively, it is imperative that every effort be made to assure that our forces are

equipped with the most effective air defense gun systems possible.

The ultimate measure of success in the design and performance of an air defense gun system is the tested ability of the system to shoot down low flying hostile aircraft within the combat environment. There exists, however, a more general need to have a measure of system effectiveness from a theoretical point of view that would allow for system evaluation and for comparative analysis of competing systems or sub-systems that could be applied in the development phase through operational fielding of the system. One such measure of effectiveness evolves from a system's ability to accurately deliver its ordnance to the target's point in space. This ability can be transformed into a kill probability based on target and gun characteristics which can be expressed as a single shot or burst kill probability.

It is this type of kill probability development as a function of gun system and target parameters that motivates this study. Computing the accuracy of a gun system and then confounding it with aircraft vulnerability data to generate a kill probability is a difficult theoretical and practical problem. There exists in the literature a profusion of formulas, tables and computational methods to assist in the solution of special cases of this type of engagement, but there is a genuine need for unifying theory within which all special cases can be rooted. Also of interest is the increasing use of simulation to model combat and predict its results.

Simulations require validated methods to compute air defense gun system kill probabilities based on prescribed battle scenarios.

The conclusion to be drawn from these comments is that the computation of system kill probability is a valuable tool which, as a current problem, merits immediate attention. It is the intent of this study to present and analyze two existing models for the computation of gun burst kill probability (BKP) in order that the general and relative merits of each might be discerned. The two models were selected for comparison because they are both in use at the present time by different agencies within the Department of the Army. One is used as a sub-model in the air defense battle simulation model, TACOS II, and the other is used by U. S. Army Weapons Command agencies for analysis of air defense gun systems in engineering development. It should be noted that this study represents the author's interpretation of these models and is not to be viewed as an official or final position report.

Model I is a model suggested for U. S. Army use by Braddock, Dunn and McDonald, Inc. [Ref. 2]. It is a relatively basic model patterned as a two-dimensional model for salvo (or burst) firing of non-fragmenting projectiles.

Model II is a model suggested for U. S. Army use by the Systems Analysis Directorate, HQ, U. S. Army Weapons Command, Rock Island, Illinois [Refs. 3 and 4]. Its development is similar to that of Model I in that target vulnerability is represented in an identical manner in each and projectile

impact points are characterized as isolated random variables. The actual development of Model II seeks to improve on Model I by being based on a less restrictive set of assumptions. The separate and aggregate effects of these assumptions are presented for comparison in this study.

The general development of Model I is traced in Chapter II of this study with more detailed attention given to the significant assumptions required in the process of model formulation. This chapter identifies the theoretical basis for the general approach of the model, and comments are added to amplify those areas believed to be most critical to the model results.

Chapter III continues with the presentation of Model II and begins the comparative analysis of the two models. Contrasts and similarities between models are presented, but with emphasis placed on the development of key aspects of Model II. Two separate cases are developed for this model after the general form of single shot kill probability (SSKP) has been shown. The two cases, A and B, contrast the effects of the assumptions regarding the magnitude of SSKP. Case A assumes that the product of the number of rounds in a burst (n) and the SSKP is small, while Case B assumes only the condition that SSKP is small.

Chapter IV concludes the comparative analysis of the two models by stating possible advantages and disadvantages of each. The general impression created by the qualitative analysis suggests that Model IIA is essentially the same as Model I, whereas Model IIB appears to be based on a less

restrictive set of assumptions than is required in either of the other developments. Quantitative results of one representative test of the three model forms are presented in this chapter to illustrate the conditions under which Model I and Model IIA diverge. Somewhat surprisingly, the quantitative analysis indicates that although Model I and Model IIB are developed with two different approaches, the BKP results remain nearly identical over the broad range of parameter values exercised in the models.

II. BRADDOCK, DUNN AND MCDONAL INC. MODEL FOR BURST KILL PROBABILITY-MODEL I

A. MODEL I INTRODUCTION

The general approach in Model I is the development of single shot kill probability for one gun system firing at one-maneuvering target which is then transformed into a burst kill probability in order that it apply more directly with the gun's normal mode of operation. This model was specifically developed to support preliminary analysis of air defense gun systems by examining the projectile/target relationships in the form of equations which describe the intercept environment. Additionally, it is used as a sub-model in the TACOS II air defense battle simulation to describe gun BKP as a function of that intercept environment.

The operation of the model involves the following steps. The gun system tracks its target, computes target characteristics, predicts a time-dependent intercept point in space, positions the gun and launches the projectile(s). The target is represented by a vulnerable area(A_v) in a plane in space that is perpendicular to the slant range(R) between the gun and the center of the symmetrically represented vulnerable area. The center of the A_v is the center of an (X,Y) coordinate system that represents the reference plane on which impact points for each projectile are measured and their distributions described. Figure 1 illustrates the geometry of the gun/projectile/target relationships.

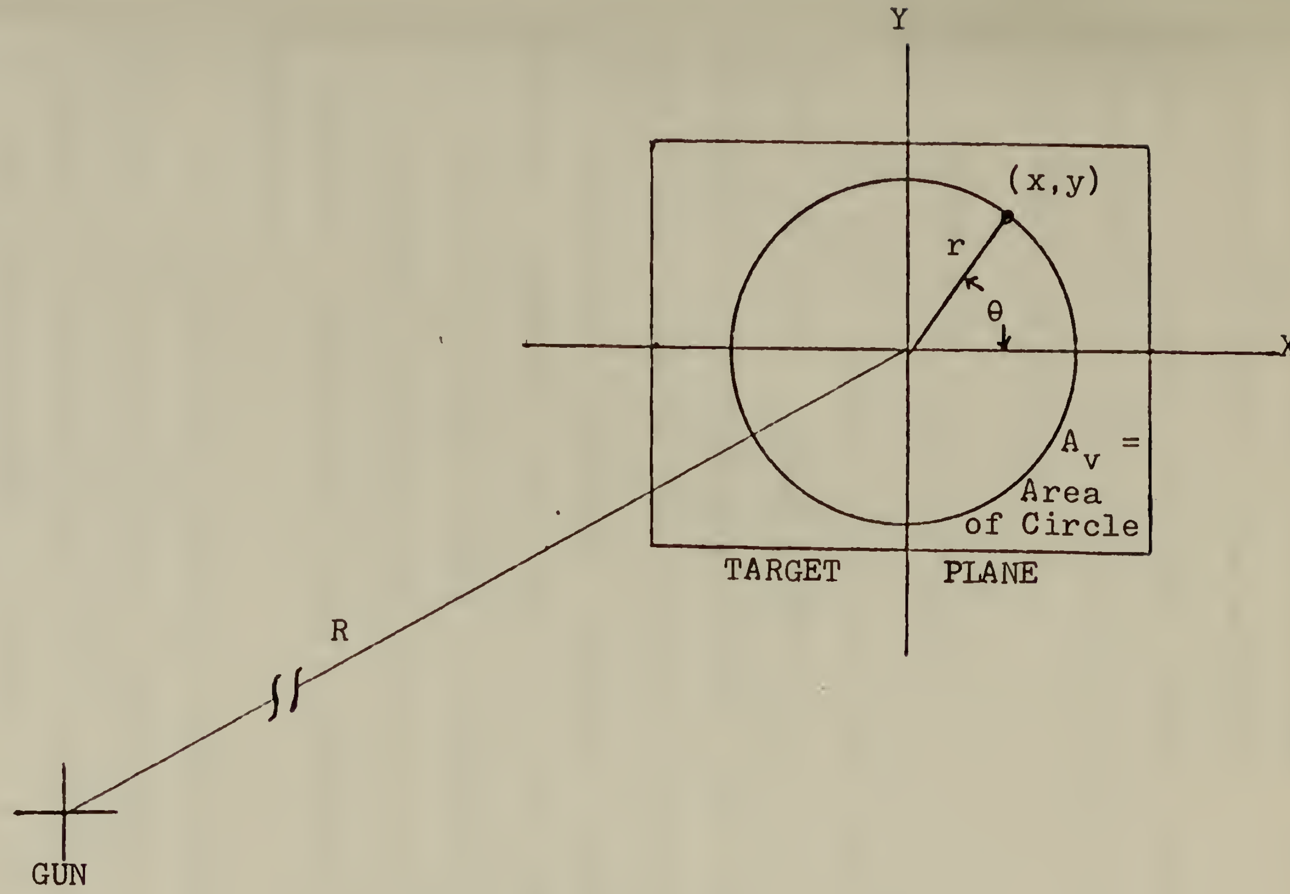


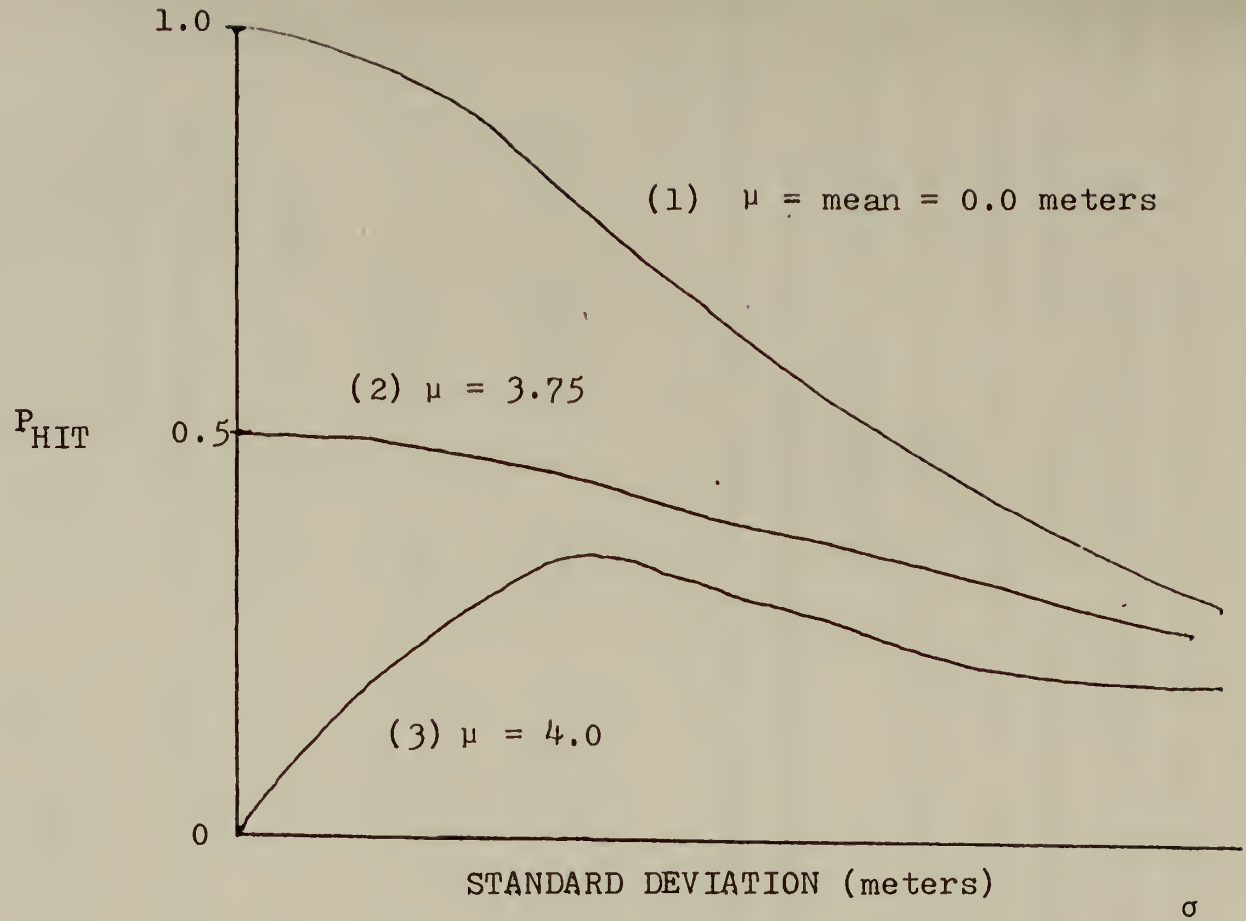
Figure 1. Model I Gun/Target Geometry

The remainder of this chapter describes the generation of the model by first developing SSKP and then transforming it into BKP salvo relying on the assumption of independence between rounds in a burst. The final form of the BKP equation is then reached through a discussion of the parameters which constitute the model.

B. SINGLE SHOT KILL PROBABILITY

The model assumes that projectile impact points in the target plane are normally distributed about each axis with zero means and independence between the random variables X and Y which describe the projectile position in that plane. (bivariate normal.) The assumption of normality relies on the Central Limit Theorem suggesting that error in the delivery of the projectile is the sum of a large number of independent error sources none of which contribute very much to the total error by themselves. Experimental data is required to validate this assumption.

The fact that the respective distributional means are assumed to be zero has special implications when considering a model for measuring single shot effectiveness of a weapon firing a non-fragmenting projectile. A non-zero value of expected impact point could result in a case where higher hit probability results from larger projectile dispersion. Figure 2 illustrates the point by depicting graphically the possible relationships among hit probability, mean impact



1-dimensional tgt

Figure 2. Hit Probability As A Function of Mean and Standard Deviation (For a 7.5 meter, one-dimensional target)

point and variance for a one-dimensional, normally distributed random variable.

Cases 1 and 2 obviously result in greatest hit probability with smallest dispersion, but Case 3 demonstrates the possible existence of a situation in which hit probability is maximized by making dispersion some finitely large value for a non-zero mean. This theory is easily expanded to two-dimensions with the same results. Reference 5 provides complete development of these cases as well as adding other insights into the effects on hit probabilities for a variety of mean, variance and distributional relationships.

The normality assumption allows the representation of delivery error as a function of two probability density functions.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\ \sigma_x} \exp\left\{-\frac{(x-\mu_X)^2}{2\sigma_x^2}\right\} \quad (1)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\ \sigma_y} \exp\left\{-\frac{(y-\mu_Y)^2}{2\sigma_y^2}\right\} \quad (2)$$

where $\mu_X = \mu_Y = 0$ $\rho = 0$ assumed since X & Y are independent

The independence of X and Y suggest the following joint density relationship:

$$P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y) = \int_{R_x} f_X(x)\, dx \cdot \int_{R_y} f_Y(y)\, dy$$

$$= \int_{R_x} \int_{R_y} \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{-\frac{1}{2}\right.$$

$$\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)\Big\} \quad dxdy \qquad (3)$$

where R_x and R_y represent the range of target values for x and y respectively.

This joint normal density representing projectile distribution (delivery error) in the target plane has utility in its present form, but it is transformed into polar coordinates since target vulnerability is represented in this model development by a circular area. It is also convenient to assume equal variances with respect to each axis in order that the joint normal might be represented as a circular normal distribution. $\sigma_x = \sigma_y = \sigma$ ⇒ circular normal

Reference 6, however, suggests that the greatest variability in projectile error occurs along the azimuth(X)-axis where the target velocity component is greatest. This implies that the variances are in fact not equatable. $\sigma_x > \sigma_y$

Continuing the transformation to polar coordinates:

$$P_{R,\theta}(r,\theta) = \int_0^{2\pi} \int_0^{r} \frac{1}{2\pi\,\sigma^2} \exp\left\{-\frac{r^2}{2\,\sigma^2}\right\} rdrd\theta \qquad (4)$$

The model assumes at this point that the target area seen from the gun may be approximated by a smaller vulnerable area (A_v) which is represented geometrically as a circle with its total area equal to the target vulnerable area. This assumption relieves a great mathematical burden in the development of BKP, but it creates another burden for the user who must

assume tgt = circle

to be estimated

$A_v = f$(tgt type, aspect & range)

develop a reasonable technique for estimating A_v for a given target type, aspect angle and range. The notion of vulnerable area is discussed further in Chapter III.B.

Continuing the development:

$A_v = \pi r^2$ such that $r = \sqrt{A_v/\pi}$

Equation (4) now becomes:

$$P_{R,\theta}(r,\theta) = \int_0^{2\pi} \int_0^{\sqrt{A_v/\pi}} \frac{1}{2\pi\sigma^2} \exp\{-\frac{r^2}{2\sigma^2}\}\, r\,dr\,d\theta \quad (5)$$

If A_v is developed such that one round impacting in that area results in a kill of the target, then: assume: 0-1 kill probability

$$SSKP = \frac{1}{2\pi\sigma^2} \int_0^{2\pi} \int_0^{\sqrt{A_v/\pi}} \exp\{-\frac{r^2}{2\sigma^2}\}\, r\,dr\,d\theta \quad (6)$$

and finally,

$$SSKP = 1 - \exp(-\frac{A_v}{2\pi\sigma^2}) \quad (7)$$

C. BURST KILL PROBABILITY

The extension of this model to account for BKP is facilitated by the assumption of total independence among rounds in a burst. It is not an intuitively appealing assumption. Guns with high rates of fire, as a minimum, suggest some correlation between rounds with the same aim point and separated in the firing sequence by only a very small increment of time.

indep. between rds

Consider the simple case of a burst of two rounds. Let A and B represent the events that each round hits the target. The first round fired corresponds to A and the second round corresponds to B. General probability theory suggests:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\text{where } P(A \cap B) = P(A) \cdot P(B/A)$$

Under the assumption of independence between rounds:

$$P(B/A) = P(B)$$

if no independence between rds & $\rho > 0$, $\Rightarrow P(B|A) > P(B)$ & $P(A \cup B) > P(A \cup B)$ for non-independent rds

and

$$P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$$

If the assumption is not valid, then the degree of correlation between A and B must be ascertained in order that its effects on the model be known. For example, if testing demonstrated that there was positive correlation between A and B, then the conditional probability of B given A would always be greater than the unconditional probability of B. The result would be a consistently higher estimate for BKP. The degree of this correlation would dictate the magnitude of the error in the general model caused by such an assumption. It remains possible, however, that the net effect of an erroneous assumption at this point might be negligible, so once again test data is required to support or refute it. Possible error notwithstanding, the development continues with n equal to the number of rounds per burst, each normally distributed and each having independent SSKP as developed.

if $\rho > 0$ $P(B|A) > P(B)$

$n = \#$ rd / burst

$r \sim N(\mu, \sigma)$

$$P(\text{all } n \text{ miss}) = (1 - SSKP)^n \tag{8}$$

$$BKP = 1 - P(\text{all } n \text{ miss}) = 1 - (1\text{-}SSKP)^n$$

from Eq. (7),

$BKP = 1 - \left(1 - \left[1 - e^{-\frac{A_v}{2\pi\sigma^2}}\right]\right)^n$

$= 1 - e^{\frac{nA_v}{2\pi\sigma^2}}$

$$BKP = 1 - \exp\left(- \frac{n\ A_v}{2\pi\ \ \sigma^2} \right) \tag{9}$$

✓

This model asserts that BKP increases with an increase in target A_v and with an increase in the number of rounds in a burst. The variance term suggests BKP decreases if the gun system is less accurate in its delivery capability. In its current form, BKP is a function of three parameters none of which causes counterintuitive impact on the value of BKP. One difficulty, however, arises when the user tries to incorporate this model into his scenario. Where does one acquire the appropriate values for n, A_v and σ^2?

Burst size can be fixed at a reasonable level based on known firing rates or it may be simply varied in the equation for BKP and in field tests to determine the degree of its effect on BKP. Vulnerable area and variance are not so easily deduced. A_v is not necessarily a function simply of target total area, but rather it is more probably a function of an aggregation of many factors relating directly to the target, and perhaps others attributed to the gun and its projectile characteristics. This issue is discussed further in the development of Model II in Chapter III.B.

The variance in a gun system's delivery capability is another factor that has considerable impact on model results, but is difficult to determine quantitatively. Expressions for variance have been advanced with varying degrees of resolution, and they all share the commonality of being just estimates. This model considers only the aggregated dispersion due to aiming and ballistic errors rather than an in depth analysis of the many system functions that contribute to the net effect. The model form allows for easy adjustment as techniques improve for developing system variation components. Variance as represented in Eq. (9) is a function of slant range (R) between gun and target as well as being related to aiming and ballistic errors. Model form requires that variance be expressed in units such as square meters. If the aiming and ballistic contributions to variance are developed in angular units, and if it is assumed that angular dispersion is constant over range, then Eq. (9) is easily modified to include the range factor.

σ = ∡ dispers. $\}\sigma$

range

$$\sigma^2(\text{meters}^2) = R^2\ (\text{meters}^2) \cdot \sigma_r^2\ (\text{radians}^2)$$

such that

$$BKP = 1 - \exp\left(- \frac{A_v\, n}{2\pi R^2 \sigma_R^2} \right) \qquad (10)$$

Consider the sources of variance in the projectile distribution. This model suggests, as does Helgert in Ref. 7, that variance may be thought of as an aggregation of independent variances due to ballistic dispersion and due to aiming errors.

Let σ_B^2 and σ_A^2 represent these variance terms in square meters. The aiming error variance represents all those factors which contribute to the condition that the gun is not actually pointing exactly at the desired aim point at the time of firing. It constitutes effects due to components that compute target parameters, system servo mechanisms, alignment problems, gunner effects and others. These factors, as was discussed in a previous section of this study, are assumed to result in a normally distributed aiming error with zero mean and the variance term as just described. In general,

$$\sigma_A^2 \text{ (meters}^2) = R^2 \text{ (meters}^2) \cdot \sigma_{AR}^2 \text{ (radians}^2)$$

Ballistic dispersion is a function of many projectile characteristics such as muzzle velocity and weight plus effects due to ambient conditions. Experience has supported the assumption that ballistic error be represented as a random variable with a normal distribution of zero mean and variance as given. In general,

$$\sigma_B^2 \text{ (meters}^2) = R^2 \text{(meters}^2) \cdot \sigma_{BR}^2 \text{ (radians}^2)$$

Considering that the sum of two independent normal random variables is again a normally distributed random variable with parameters also additive, it is possible to conclude that:

$$\sigma_r^2 \text{ (radians}^2) = \sigma_{AR}^2 + \sigma_{BR}^2 \text{ (radians}^2)$$

When applied to Eq. (10), the result is the final form for Model I.

$$BKP = 1 - \exp\{ - \frac{n \cdot A_v}{2\pi R^2(\sigma_{AR}^2 + \sigma_{BR}^2)} \} \tag{11}$$

final result for BDM model

III. U. S. ARMY WEAPONS COMMAND MODEL FOR BURST KILL PROBABILITY-MODEL II

A. MODEL II INTRODUCTION

Model II's development parallels that of Model I in that it first develops an expression for single shot kill probability and then transforms that into an appropriate form for burst kill probability. It constitutes an expansion of the Analytic Gun Model developed by the University of Michigan, System Research Laboratory for the U. S. Army Weapons Commnd. See Ref. 8 for the original development.

The model asserts that target vulnerability can be represented in a plane perpendicular to the slant range from the gun to the target at the point of predicted intercept. There is existing methodology that would allow the representation of three-dimensional targets, but its adaptation at this point in the model would be unnecessarily cumbersome to its development. Reference 8 describes the methodology as well as amplifies the notion of vulnerable area.

The projectile distribution in this target plane has a probability density function defined as $f_{X,Y}(x,y)$ where the x and y values relate to coordinate axes centered at the target center of vulnerability. Some analyses refer to this origin as intended aim point. This density describes the position of the round at the predicted time of intercept(t).

Part B of this chapter initiates the theoretical basis for this model by describing the development of single shot kill probability. The distributional form of the random variables that describe rounds impacting the target plane is proposed and the notion of vulnerable area is addressed in detail. Once SSKP is presented, the concept of a burst center random variable is introduced. This concept is one method for improving upon the undesirable assumption of independence between rounds in a burst used in Model I to transform SSKP to BKP. It is at this stage of the development of BKP in Part C of this chapter that Model II is divided into two cases. The two cases, A and B, are necessary in order that a key assumption regarding the magnitude of SSKP might be approached from two different aspects. The development of each case is related with some final comments concerning contrasts in the theoretical bases for Model I and the two cases of Model II.

B. SINGLE SHOT KILL PROBABILITY

The general development of SSKP relies on several key assumptions. The first of which is that target vulnerable area remains constant over the time of the engagement, i.e. $A_v(t) = A_v$. Target/gun geometry does not allow this to be technically true, but the validity of BKP as a good measure of system effectiveness is not necessarily negated by its acceptance. Assume further that the total aircraft silhouetted area (A_t) can be systematically transformed into a representative

vulnerable area, and that one hit in that area produces a target kill. If the distribution of the projectile in the target plane has a large dispersion with respect to that area, then the small target approximation may be invoked to conclude that:

$$SSKP \approx f_{X,Y}(x,y) \cdot A_v$$

small tgt appx

where x and y are evaluated at target center (0,0).

The small target approximation is noted here even though the necessary distributional assumptions for the projectiles have not been advanced. It is known that this model will assume a bivariate normal distribution for the projectiles in the target plane with zero means and no correlation. The result is a distribution of the form hypothesized in Model I and represented by Eq. (3). If the variances are large relative to the x and y values for target dimensions, then the quotients in the power of the exponential are approximated by zero and the double integral reduces to:

$$\frac{1}{2\pi\,\sigma_x\sigma_y} \cdot x \cdot y$$

$$= \int_{R_x}\int_{R_y} \frac{1}{2\pi\sigma_x\sigma_y} e^{-0}\,dx\,dy$$

where $x \cdot y$ = target area for a rectangular target which equates to A_v. The quotient in this approximation corresponds to the density function in the SSKP formula when it can be assumed normal with zero means, no correlation and evaluated at the aim point, (0,0).

Considering this approximation as an analog to an argument of conditional probability may help to clarify its basis while at the same time providing some insight to the notion of vulnerable area. In general,

$$P(\text{kill}) = P(\text{hit}) \cdot P(\text{kill/hit})$$

which when applied to this example suggests:

$$P(\text{kill}) = P(\text{hit on } A_t) \cdot \frac{A_v}{A_t}$$

where the quotient here is essentially a lethality function in which the assumption that A_v is uniformly distributed over A_t is implicit. The quantification of A_v is a potential problem for the user of the model. Ballistic Research Laboratories are said to provide data of this nature, but in order to effectively exercise the model, one must be aware of the subtleties involved in defining A_v. The analogy to conditional probability suggests that:

$$P(\text{hit on } A_t) = f_{X,Y}(x,y) \cdot A_t$$

where it can be seen that the density function is the probability of hitting a vulnerable area of unit size. This factor is based exclusively on gun system characteristics and leads to a slightly modified perspective of the original equation for SSKP. Now,

$$SSKP \approx f_{X,Y}(x,y) \cdot A_t \cdot \frac{A_v}{A_t}$$

so that in the original form for SSKP, A_v equates to the product of A_t and the ratio of A_v to A_t. This term provides

input for the effects due to target parameters as well as some consideration for projectile characteristics.

The conclusion to be drawn from this is that SSKP in its present form may be computed by describing during the course of an engagement the presented target area (A_T), the lethality relationship for the given gun and target (f_V/A_T) and the distribution of projectiles ($f_{x,y}(x,y)$) over the target A_V.

$SSKP = f_{xy}(x,y)A_v = f_{xy}(x,y)A_T \cdot \frac{A_v}{A_T}$

The user is still somewhat burdened at this point by the need to determine a reasonable lethality relationship. Model I assumes the simplest case of uniformly distributed vulnerability over total target area. If the target is depicted geometrically as a shoebox in space, then the gun system views a silhouetted area made up of three side-related components of the box. The target vulnerable area is then just a scaled down representation of that total area as illustrated in Figure 3. The mathematics of the development is facilitated by the conversion of A_V to a circular area of equivalent magnitude. Parry, in Ref. 9, suggests a higher resolution methodology for determining A_V as a function of gun/target aspect angle and total presented area. It assumes pockets with varying degrees of vulnerability within the target profile as illustrated in Figure 4. The effect of a round impacting in the cockpit or engine compartment might produce more catastrophic consequences than a round impacting in an avionics section or some other less critical area. The notion of compartments of vulnerability that may be aggregated to a single vulnerability

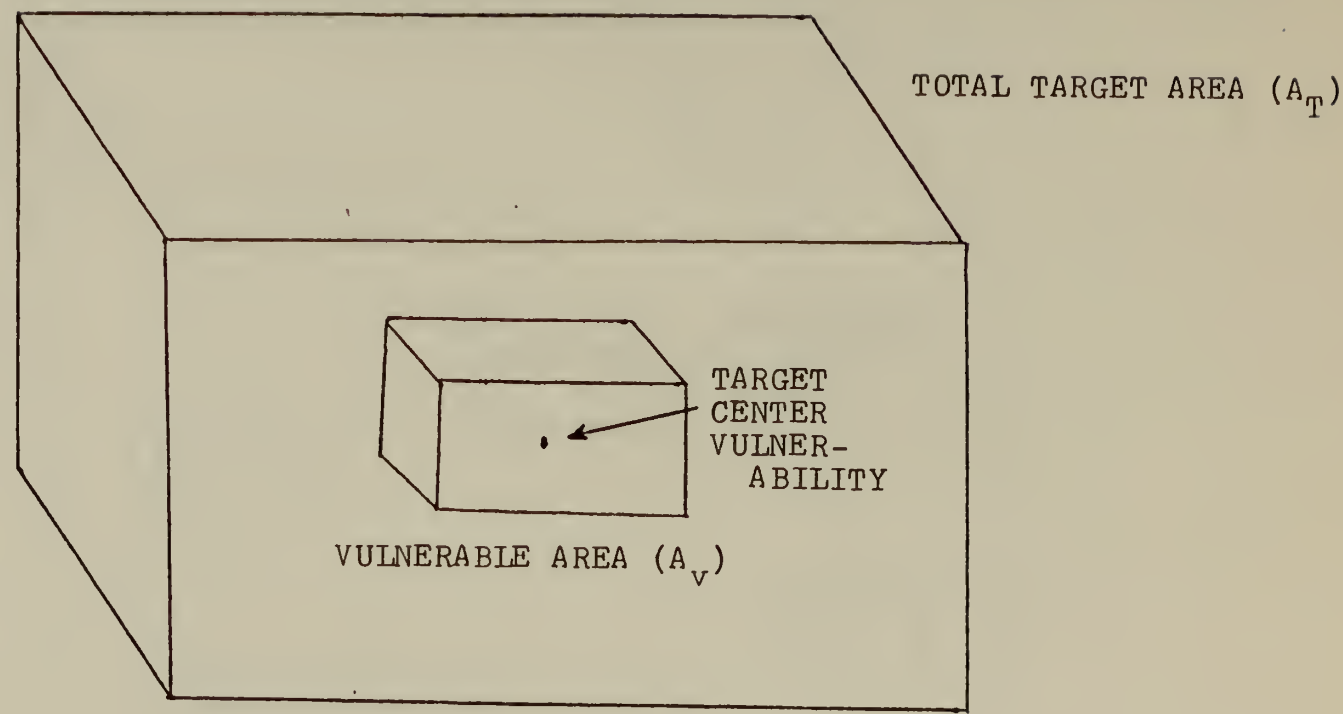


Figure 3. Shoebox Representation of a target in Space (with uniformly distributed vulnerability)

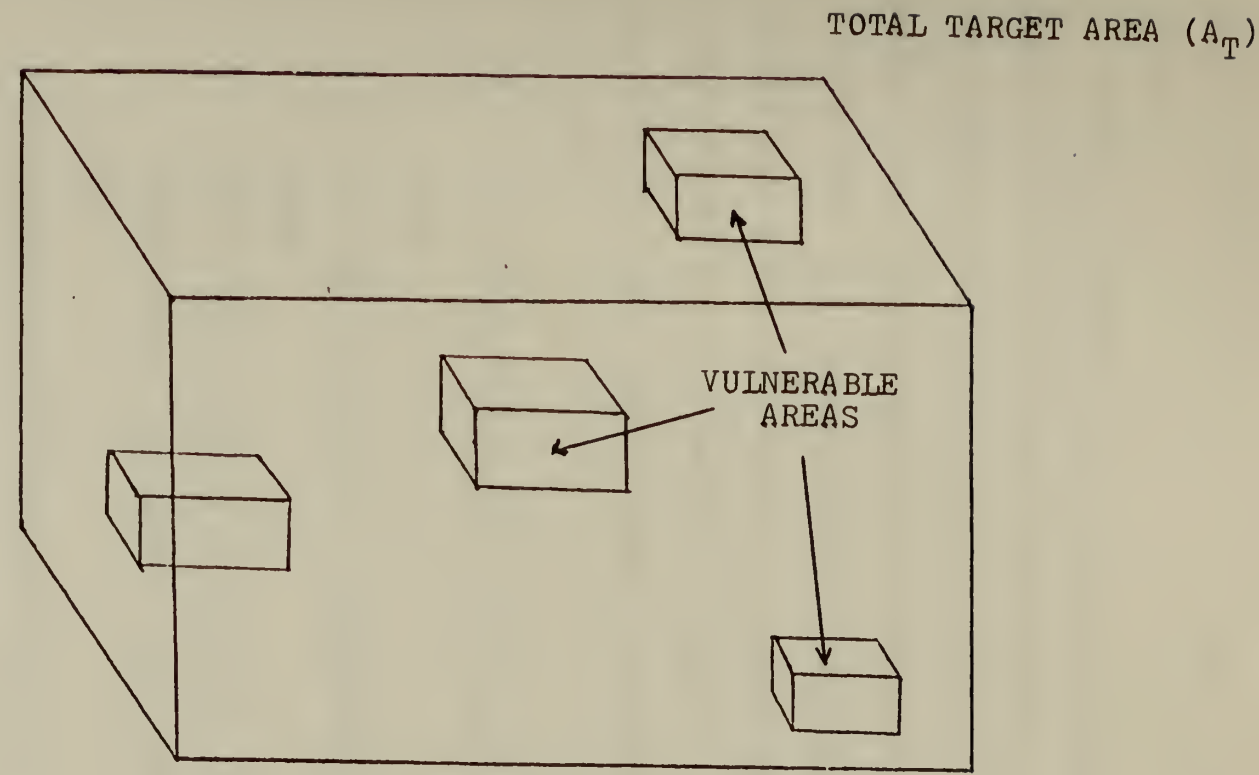


Figure 4. Shoebox Target in Space with Compartments of Vulnerability

factor has its merits in theory and in practice may be applied to this model just as simply as less detailed techniques.

Having considered the notion of vulnerable area, it remains for this chapter to develop an acceptable form for (density) the density describing the projectile distribution in the target plane before the final form for SSKP can be presented.

Consider the locations in three-space (R_3) of both the target and the projectile at the predicted time of intercept. A three-dimensional coordinate system is proposed in R_3 with its origin at the target center of vulnerability with axes X, Y, and Z.

X = horizontal (azimuth) coordinate in the plane of the target

Y = vertical (elevation) coordinate in the plane of the target

Z = range coordinate perpendicular to the plane of the target along the slant range from the target to the gun.

It is assumed at the time of predicted intercept (t) that:

azimuth elevation, range

(i) There exists an error vector $(\Delta A, \Delta E, \Delta R)(t)$ corresponding to the distance between the target center of vulnerability and the projectile.

(ii) Each component of the error vector is the sum of random errors due to independent random variables, e.g. $\Delta A = \sum_{i=1}^{n} (\Delta A)i$. The Central

Limit Theorem is invoked such that the error vector may be assumed to have a multivariate normal distribution of dimension three and zero covariances.

This is a major departure from conditions described in Model I, in that due to range error, ΔR, the projectile will probably not be coincident with the target plane at (t). Consequently, the error distribution in R_3 at (t) must be projected into the actual target plane R_2. The projectile distribution in R_2 corresponds to the multivariate normal distribution in R_3 at a time when range-to-target and range-to-projectile are equivalent for the gun. Define a new set of coordinate axes in R_2 as (X', Y') which correspond to the X and Y axes in R_3, respectively.

At the time of predicted intercept, the target is described by the velocity components $(\dot{A}\cdot R, \dot{E}\cdot R, \dot{R})(t)$, and the projectile velocity components are $(0, 0, V_p)(t)$, where:

$\dot{A}$ = target velocity in the X-direction in radians/second

$\dot{E}$ = target velocity in the Y-direction in radians/second

$\dot{R}$ = target velocity in the Z-direction in meters/second

R = slant range from gun to target in meters

V_p = projectile velocity in meters/second

The non-maneuvering target is assumed to have a constant velocity vector over the time of flight, t_f, of the projectiles. The degree of effect of this assumption can be substantial if the target is changing velocity during the engagement,

but there is no effect on the model as long as it is understood that this model is restricted to the non-maneuvering, constant velocity target case. Theory has been advanced for the maneuvering target case in Ref. 4, but it will not be addressed in this study.

The time between conditions of equal range and time of predicted intercept is represented by the random variable, Δt. It is easily assumed that Δt is small such that the projectile velocity, V_p, may be considered constant over Δt, and the relative velocity of the projectile with respect to the target, $(\dot{R}(t) + V_p(t))$, may be approximated by $V_p(t) = V_p$. Therefore,

$$\Delta t \simeq - \frac{\Delta R}{V_p}$$

Returning to the error vector $(\Delta A, \Delta E, \Delta R)(t)$, the multivariate normal assumption implies that:

$$\Delta A \sim N(M_x, \sigma_x^2)$$

$$\Delta E \sim N(M_y, \sigma_y^2)$$

$$\Delta R \sim N(M_z, \sigma_z^2)$$

These random variables are projected into R_2 at $t + \Delta t$. This projection results in the creation of two new random variables $\Delta A'$ and $\Delta E'$ which relate azimuth and elevation errors to the X' and Y' axes, respectively. Considering Eq.(13), the error terms become:

$$\Delta A'(t + \Delta t) = (\Delta A - \Delta t \cdot \dot{A} \cdot R)(t)$$

$$= (\Delta A + \frac{\Delta R}{V_p} \cdot \dot{A} \cdot R)(t) \qquad (14)$$

$$\Delta E'(t + \Delta t) = (\Delta E - \Delta t \cdot \dot{E} \cdot R)(t)$$

$$= (\Delta E + \frac{\Delta R}{V_p} \cdot \dot{A} \cdot R)(t) \qquad (15)$$

$\Delta A'$ and $\Delta E'$ are the sums of the independent, normal random variables ΔA, ΔE, and ΔR. Therefore, they have a bivariate normal distribution with possible correlation and with parameters given by:

$$\Delta A \sim N(M_x', \sigma^2_{x'})$$

$$\Delta E \sim N(M_y', \sigma^2_{y'})$$

The result of combining these two distributions is that at $t + \Delta t$,

$$f_{X',Y'}(x,y) = \frac{1}{2\pi\,\sigma_{x'}\sigma_{y'}\sqrt{1-\rho^2}} \cdot \exp\left\{\frac{1}{-2(1-\rho^2)}\right.$$

$$\left\{\frac{(x-M_{x'})^2}{\sigma^2_{x'}} - 2\rho\frac{(x-M_{x'})(y-M_{y'})}{\sigma_{x'}\ \sigma_{y'}} + \frac{(y-M_{y'})^2}{\sigma^2_{y'}}\right\}\left.\right\} \qquad (16)$$

This function must be evaluated at some point in the target plane in order that it represent the engagement at intercept. It is assumed that the target is 'perfectly'

located and that the center of vulnerability is in fact the aim point so that the projectile distribution is projected into the target plane with the effect measured over target center. Therefore, the function is evaluated at $(x,y) = (0,0)$. When combined with the vulnerable area term, this yields the final form for SSKP:

result

$$SSKP = A_v \cdot f_{X',Y'}(0,0)$$

$$= \frac{A_v}{2\pi\,\sigma_{x'}\sigma_{y'}\sqrt{1-\rho^2}} \exp\left\{ \frac{1}{-2(1-\rho^2)} \left\{ \left(\frac{M_{x'}}{\sigma_{x'}}\right)^2 - 2\rho \frac{M_{x'}}{\sigma_{x'}} \frac{M_{y'}}{\sigma_{y'}} + \left(\frac{M_{y'}}{\sigma_{y'}}\right)^2 \right\} \right\} \qquad (17)$$

C. BURST KILL PROBABILITY

Model I assumed at this point in its development that rounds in a burst are independent of each other in effects on the target. As was discussed, this is counterintuitive, but it has merit in that it conveniently eliminated some formidable obstacles in the model development. Model II suggests that the negative aspects of this independence assumption may be lessened by considering an alternative solution to the transformation from SSKP to BKP.

The alternative proposes that groupings of rounds that are either distinctly separated bursts or arbitrarily divided bursts be considered for collective effect. The centers of the bursts are defined as (R_1,R_2) and are assumed to be

independent, normally distributed random variables. The advantage to be gained from the introduction of this random variable is that the Central Limit Theorem confirms that the distribution of the burst centers is in fact normal regardless of the underlying distribution of the individual projectiles. By assuming the mean value of the projectile distribution to be the burst center, BKP may be developed by conditioning on the known burst center distribution.

The following terms are defined to assist in the development of the BKP equation:

SSMP = Single Shot Miss Probability = 1 - SSKP

BMP = Burst Miss Probability = 1 - BKP

n = number of rounds per burst

(R_1, R_2) = burst center coordinates in the (X', Y') plane

(X_1, X_2) = position of the individual projectiles impacting the (X', Y') plane[1]

Assume that:

(i) X_1 and X_2 are independently distributed in the (X', Y') plane. This facilitates bypassing the problems associated with trying to quantify the correlation between the two random variables. The mathematics which follow are also more tractable. Once again, it is difficult to assess the ultimate

---

[1]This is identical with the (X,Y) notation used earlier. It is changed here to facilitate the matrix notation that will follow in the solution.

effect on the model results, but it is clear that increasing the number of assumptions of this type is narrowing the margin of difference between the developments of these two models. This particular assumption implies:

$$X_i \sim N(R_i, \sigma_i^2) \qquad i = 1,2.$$ individual rds

(ii) Burst centers (R_1,R_2) are normally distributed with:

$$R_1 \sim N(M_3, \sigma_3^2)$$

burst centres

$$R_2 \sim N(M_4, \sigma_4^2)$$

with correlation coefficient, ρ .

σ_1^2 and σ_2^2 are dispersion factors in the distribution of individual rounds due to ballistic errors and gun dynamics errors. The σ_3^2 and σ_4^2 variance terms are a measure of the dispersion of the burst centers in the target plane. The subscripts 1 and 2 replace x' and y' for convenience. Consider the development of BKP conditioned on the distribution of the burst centers.

$$BMP = E(BMP/(R_1,R_2)) \tag{18}$$

$$= E((SSMP)^n/(R_1,R_2))$$

$$= E((1\text{-}SSKP)^n/(R_1,R_2)) \tag{19}$$

assumption for case I :

If $n \cdot SSKP$ is small, then $(1\text{-}SSKP)^n$ may be approximated by the first two terms of its binomial expansion. This assumption essentially says that the gun system being considered is functionally incapable of performing its intended mission. For gun systems with high rates of fire, experience suggests that SSKP may be assumed small, but not $n \cdot SSKP$. Development of the model under both of these conditional assumptions are traced here to illustrate the impact of each.

First, if $n \cdot SSKP$ is small, Eq. (19) may be expressed as:

$$BMP = E\,\{(1-n \cdot SSKP)/(R_1, R_2)\}$$

$$= E\,\left\{1 - \frac{n \cdot A_v}{2\pi\,\sigma_1\,\sigma_2} \cdot \exp\left\{-\frac{R_1^2}{2\sigma_1^2} - \frac{R_2^2}{2\,\sigma_2^2}\right\} \Big/ (R_1, R_2)\right\}$$

Therefore,

$$BKP = E\left\{\frac{n \cdot A_v}{2\pi\,\sigma_1\,\sigma_2} \cdot \exp\left\{-\frac{R_1^2}{2\,\sigma_1^2} - \frac{R_2^2}{2\,\sigma_2^2}\right\} \Big/ (R_1, R_2)\right\} \quad (20)$$

The solution of this expectation is facilitated by the use of matrix notation. Let:

$$V = \begin{bmatrix} \sigma_3^2 & \rho\,\sigma_3\,\sigma_4 \\ \rho\,\sigma_3\,\sigma_4 & \sigma_4^2 \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \quad \text{and} \quad \underline{M} = \begin{bmatrix} M_3 \\ M_4 \end{bmatrix}$$

Consider the following solution to a general conditional expectation as a model for the form of BKP as it is transformed from Eq. (20) to Eq. (21). From Ref. 11,

$$E\{X\} = \int_{-\infty}^{\infty} E\{X/Y=y\} \cdot f_Y(y)\, dy$$

Therefore,

$$BKP = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{n \cdot A_v}{2\pi\ \sigma_1\ \sigma_2} \cdot \exp\left(-\frac{1}{2}\ \underline{R}' \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\sigma_2^2} \end{bmatrix} \underline{R}\ \right) \cdot$$

$$\cdot \frac{1}{2\pi \left| V \right|^{\frac{1}{2}}} \cdot \exp\left(-\frac{1}{2}\ (\underline{R} - \underline{M})'\ V^{-1}(\underline{R}-\underline{M})\right)\ dR_1\ dR_2 \tag{21}$$

Banash, in Ref. 3, develops the solution to this double integral by manipulating its terms until it can be observed that it is the integral form of the moment generating function of a multivariate normal distribution such that:

$$BKP = \frac{n \cdot A_v}{2\pi\ \sigma_1\ \sigma_2\ \left| I+A \right|^{\frac{1}{2}}} \cdot \exp\left(-\frac{1}{2}\ \underline{\nu}'A\ (\ I-(I+A)^{-1}A)\underline{\nu}'\right) \tag{22}$$

$$\text{where } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \sigma_3^2/\sigma_1^2 & 0 \\ 0 & \sigma_4^2/\sigma_2^2 \end{bmatrix}$$

and $\underline{\nu} = \underline{0}$, if it is assumed that burst center means are zero.

The result is that:

$$BKP = \frac{n \cdot A_v}{2\pi\sigma_1\ \sigma_2\ \left| I+A \right|^{\frac{1}{2}}} \tag{23}$$

$$\text{where } \left| I+A \right|^{\frac{1}{2}} = \begin{vmatrix} 1+\ \sigma_3^2/\sigma_1^2 & 0 \\ 0 & 1+\ \sigma_4^2/\sigma_2^2 \end{vmatrix}^{1/2}$$

$$= \frac{1}{\sigma_1 \cdot \sigma_2} \left\{ (\sigma_3^2 + \sigma_1^2) \cdot (\sigma_4^2 + \sigma_2^2) \right\}^{\frac{1}{2}}$$

and the final form for Model IIA,

result

when (n.SSKP) is small

$$BKP = \frac{n\cdot A_v}{2\pi\left((\sigma_3^2+\sigma_1^2)\cdot(\sigma_4^2+\sigma_2^2)\right)^{\frac{1}{2}}} \tag{24}$$

Now consider the second case where only SSKP is considered small. The development proceeds essentially the same as that of the first case, by conditioning on the location of burst centers.

$$BMP = E(\ (1-SSKP)^n/(R_1,R_2)\ )$$

Consider the expansion of $(1-SSKP)^n$.

$$(1-SSKP)^n = 1-n\cdot SSKP + \frac{n(n-1)}{2!}\cdot SSKP^2$$

$$- \frac{n(n-1)(n-2)}{3!}\cdot SSKP^3 + ----$$

$$= \sum_{j=0}^{n} \binom{n}{j}(-SSKP)^j$$

Then,

$$BMP = E\{\sum_{j=0}^{n} \binom{n}{j} \ (-SSKP)^j/\ (R_1,R_2)\}$$

$$= \sum_{j=0}^{n} E\{\ \binom{n}{j}\ (-SSKP)^j/(R_1,R_2)\}$$

$$= \sum_{j=0}^{n} \binom{n}{j}\ (-1)^j\ \left\{\frac{A_v}{2\pi\ \sigma_1\ \sigma_2}\right\}^j\ \left|I+A_j\right|^{-\frac{1}{2}}$$

$$\cdot\exp\left(-\tfrac{1}{2}\ \underline{\nu}'A_j(I-(I+A_j)^{-1}A_j)\ \underline{\nu}\ \right)$$

Banash shows in Ref. 4 that $A_j = j \cdot A$, such that:

$$BKP = 1\text{-}BMP$$

$$= \sum_{j=1}^{n} \binom{n}{j} (-1)^{j+1} \left\{ \frac{A_v}{2\pi\sigma_1\sigma_2} \right\}^j \left| I + j \cdot A \right|^{-\frac{1}{2}}$$

$$\cdot \exp\left(-\tfrac{1}{2} \cdot j \, \underline{\nu}' A (I - j(I + jA)^{-1} A) \underline{\nu}\right) \tag{25}$$

BKP is difficult to quantify in this form, however, it may be greatly simplified if the burst means once again are assumed zero and if the correlation between R_1 and R_2 is small enough to assume to be zero.

The result of combining these assumptions with the general BKP formula is:

$$BKP = \sum_{j=1}^{n} \binom{n}{j} (-1)^{j+1} \left\{ \frac{A_v}{2\pi\sigma_1\sigma_2} \right\}^j \left| I + jA \right|^{-\frac{1}{2}} \tag{26}$$

where

$$\left| I + jA \right|^{-\frac{1}{2}} = \frac{\sigma_1\sigma_2}{\{(\sigma_1^2 + j\sigma_3^2)(\sigma_2^2 + j\sigma_4^2)\}^{\frac{1}{2}}}$$

Adding this result to Eq. (26) produces the final form of Model IIB:

result for SSKP only small

$$\underline{BKP} = \sum_{j=1}^{n} \binom{n}{j} (-1)^{j+1} \left\{ \frac{A_v}{2\pi\sigma_1\sigma_2} \right\}^j \cdot \frac{\sigma_1\sigma_2}{\{(\sigma_1^2 + j\sigma_3^2)(\sigma_2^2 + j\sigma_4^2)\}^{\frac{1}{2}}} \tag{27}$$

When just one round is fired, this result for BKP is exactly the same as the result in Eq. (24) when it was assumed that n·SSKP was small rather than just SSKP. This

adds some degree of confidence to the mathematical manipulations involved in reaching the two final forms of the BKP equations.

It is interesting to note at this point that although the developments of Models I and II have appeared to follow completely different tacks, it seems that Model I and Model IIA have produced the same result. It is perhaps more clear if one considers the following. In general, for small z,

$$1 - e^{-z} \approx 1-(1-z) = z$$

Therefore, from Model I, considering the original assumption that dispersion was much larger than vulnerable area, Eq. (11) may be restated as:

$$\text{BKP} \approx \frac{n \cdot A_v}{2\pi \; (R^2(\sigma_{AR}^2 + \sigma_{BR}^2))} \qquad (28)$$

This result may be viewed as nearly identical with Model IIA as expressed in Eq. (24). The only possible difference being the manner of generating and expressing the variance components of projectile delivery. Thus, it might be deduced that Model I is implicitly assuming in its development that n x SSKP is small, or conversely, Model IIA may have implicit within its development the assumption of independence between rounds of a burst. The negative significance of these two basic assumptions cast serious doubt on the veracity of each of these models as they exist in Eq. (11) and Eq. (24).

Nonetheless, the extension of Model II resulting in Eq. (27) provides alternatives to these two undesirable assumptions. Stating optimism for this modification to the model is one thing, validating its results remains another. Some discussion has been presented regarding the notion of vulnerable area since it is a key input parameter to the models, but very little information has surfaced in this model about the origins of the variance terms. The Appendix is included to demonstrate the degree of resolution possible in computing variance and the manner of application to the general BKP form.

Chapters II and III of this study have traced the major steps in the development of the three model forms. Although some comparative comments have been made, it remains for Chapter IV to present the general overview of the models and to identify the salient features that might order the relative worth of each. Hypothesized examples are proposed to allow some objective interpretation of the models that has been lacking due to the subjective nature of the material presented to this point.

IV. COMPARATIVE ANALYSIS AND RESULTS

The development of both models for determining gun system BKP has demonstrated many of the usual obstacles inherent with attempts to explain mathematically complex phenomena such as military combat operations. The use of assumptions to overcome points of resistance or excessive complexity serves the function of lubrication in the development, but it is always a lubricant with a price. The extent of that cost is difficult effect to measure, especially when many assumptions are layered within the development. These two BKP models are fraught with assumptions that facilitate the mathematics of each, but which are at best difficult to evaluate with respect to the net effects on the accuracy of model results. It is for this reason that this study makes frequent reference to the need for data from actual system tests to assist in validating the models and to measure the effects of specific assumptions. Data from tests of existing systems may be applied generally to these models without the models losing their general applicability which earlier was proposed as one of their basic raisons d'etre. This study has been conducted without the benefit of test data so that assumptions have been necessarily discussed from a theoretical and intuitive point of view with conclusions withheld in many cases pending validation.

Both BKP models are initiated with the realization that BKP is one measure of effectiveness that has a broad base of application in the field of air defense gun system evaluation and simulation. They present and describe identical gun/target environments and engagement procedures whereby one gun system encounters one non-maneuvering target. The concepts involved in the process of developing both models are essentially the same as for any general target destruction model. The process consists of factors which affect the distribution of projectile impact points and the lethality of the projectile as a function of impact points and target characteristics. The factors which affect the impact point distribution may be considered as potential error producers which may be synthesized from sub-components, and the system lethality may be expressed as a function of target and projectile parameters.

These concepts allow the general approach of analysis to be one of modeling a random process. The random variables in the process are target location error, aim point error, and gun system projectile delivery error. In both models, target location errors and aim point errors have been combined for convenience and each error term is assumed to have its distribution described by some probability density function. Specifically, the circular normal and the bivariate normal distributions have been assumed appropriate for Model's I and II , respectively. When firing bursts, the aim point error is realized only once for each burst whereas the delivery error is realized for every round. This delivery error suggests

some correlation between rounds in a burst and some correlation between effects of each round when transforming SSKP to BKP.

Conclusions about a preferred model for BKP would be somewhat dangerous and suspect at this point without test data to support them, but it is possible to represent trends in the general application of the three BKP forms. Sample computations have been generated by permuting hypothesized values for burst size with varied levels of dispersion for the projectile and burst center distributions. Bursts range in size from 1 to 100 rounds while vulnerable area is held constant at 1.0 square meter. Dispersion, as represented by the standard deviations of the distributions discussed, are considered in three categories to reflect near, intermediate and long range effects. The intermediate range values for standard deviation are hypothesized and are then halved and doubled to reflect near and long range conditions, respectively. The selection of actual magnitudes for these values is completely arbitrary and in no way reflects real data. The range of values selected is intended only to provide a large enough spread to be able to identify representative trends in the model quantitative results. Tables I, II and III are presented to demonstrate the effects on BKP of these permutations, and to show trends in the models individually and comparatively.

Results for each of the three BKP forms are remarkably similar for burst sizes less than 20 in the intermediate range. The rapid increase in values for Model IIA as burst

TABLE I

Burst Kill Probabilities for Intermediate Range Target Engagements

$A_v = 1.0$ meters^2 $\sigma_1 = 4.0$ meters $\sigma_2 = 3.0$ meters $\sigma_3 = 3.0$ meters $\sigma_4 = 2.0$ meters

Burst Size	Model I	Model IIA	Model IIB
1	.008789	.008829	.008829
10	.084498	.088288	.084442
20	.161857	.176567	.160898
50	.356875	.441416	.349978
75	.484246	.662125	.470648
100	.586390	.882833	.565908

TABLE II

Burst Kill Probabilities for Near Range Target Engagements

A_v = 1.0 meters^2 σ_1 = 2.0 meters σ_2 = 1.5 meters σ_3 = 1.5 meters σ_4 = 1.0 meters

Burst Size	Model I	Model IIA	Model IIB
1	.034697	.035313	.035313
10	.297516	.353133	.396547
20	.506516	.706266	.496215
50	.828927	1.76....	.792627
75	.929243	2.64....	.887264
100	.970734	3.53....	.931872

TABEL III

Burst Kill Probabilities for Long Range Target Engagements

$A_v = 1.0$ meters^2 $\sigma_1 = 8.0$ meters $\sigma_2 = 6.0$ meters $\sigma_3 = 6.0$ meters $\sigma_4 = 4.0$ meters

Burst Size	Model I	Model IIA	Model IIB
1	.002204	.002207	.002207
10	.021829	.022071	.021826
20	.043182	.044142	.043115
50	.104483	.110354	.103908
75	.152556	.165531	.151239
100	.198049	.220708	.195749

sizes increase past 20 reflect the point at which the assumption that n x SSKP is small tends to lose its veracity. The near and long range cases support the conclusion that 'small' refers to values of approximately 0.2 or less for n x SSKP.

The development of Model IIB demonstrated what appeared to be some clear improvements theoretically over some of the more undesirable assumptions of Model I, but the sample results in each of the three tables show near equivalence in BKP values throughout. It was suggested earlier that if a positive correlation existed between rounds in a burst, then the independence assumption in Model I would result in consistently high estimates for BKP. The result of slightly lower BKP values for Model IIB may reflect the degree of affect caused by that different approach.

The point has been emphasized that the models require validation before their individual merits can be ascertained, however, they do have considerable intuitive and practical appeal as they currently exist. The relationship between model parameters and solution behavior are easily recognized in each model and they are easily applied in simulation. They represent a convenient practical and theoretical base from which to study or perform parametric analysis on existing systems or on systems still on the drawing board.

APPENDIX

Variance of the Critical Random Variables

Each component of the error vector at time, t, was assumed to be the aggregation of errors attributable to independent sources, each of which was said to be normally distributed. Consider the development of these random variables.

1. Errors Contributing to ΔA.

   a. Azimuth Sensing - Let A1(t) be the random variable denoting the angular error resulting from sensor azimuth errors. It results in error on the X-axis equal to $(A1 \cdot R)(t)$. Where,

   $$E\ \{(A1 \cdot R)(t)\} = R \cdot M_{A1}(t)$$

   $$V\ \{(A1 \cdot R)(t)\} = R^2 \cdot \sigma^2_{A1}(t)$$

   b. Azimuth Rate Sensing - Let $\dot{A}(t)$ be the random variable denoting the angular error resulting from sensor azimuth rate errors. It results in error on the X-axis equal to $(\dot{A} \cdot t_f \cdot R)(t)$ where t_f is the time of flight of the projectile to predicted intercept. Mean and variance terms are given by:

   $$E\ \{(\dot{A} \cdot t_f \cdot R)(t)\} = 0$$

   $$V\ \{(\dot{A} \cdot t_f \cdot R)(t)\} = t_f^2 \cdot R^2 \cdot \sigma^2_{\dot{A}}\ (t)$$

c. Gun Pointing Error - Let A2(t) be the random variable denoting azimuth error resulting from gun pointing errors. It results in error on the X-axis equal to (A2·R)(t). Mean and variance are given by:

$$E\{(A2 \cdot R)(t)\} = R \cdot M_{A2}(t)$$

$$V\{(A2 \cdot R)(t)\} = R^2 \cdot \sigma^2_{A2}(t)$$

2. Errors Contributing to ΔE.

Errors with respect to the Y-axis which constitute $\Delta E(t)$ are developed identically as for $\Delta A(t)$.

3. Errors Contributing to ΔR.

a. Range Sensing - Let R1 be the random variable denoting the range error resulting from range sensor errors. It results in error on the Z-axix equal to (R1)(t). Mean and variance are given by:

$$E\{(R1)(t)\} = M_{R1}(t)$$

$$V\{(R1)(t)\} = \sigma^2_{R1}(t)$$

b. Range Rate Sensing - Let R be the random variable denoting range error resulting from range rate sensing errors. It results in an error on the Z-axis equal to $(\dot{R} \cdot t_f)(t)$. Mean and variance terms are:

$$E\{(\dot{R}\cdot t_f)(t)\} = t_f \cdot M_{\dot{R}}(t)$$

$$V\{(\dot{R}\cdot t_f)(t)\} = t_f^2 \cdot \sigma_{\dot{R}}^2(t)$$

c. Muzzle Velocity Error - Let ΔV be the random variable denoting muzzle velocity variations resulting in range errors (ΔR). Reference 1, by employing the 3/2 power law, shows that the range error contributions may be represented as:

$$E\{\Delta R\} \text{ due to } \Delta V = 0$$

$$V\{\Delta R\} \text{ due to } \Delta V = \left(\frac{R}{V_p}\right)^2 \cdot \sigma_V^2$$

Considering these error contributions random variables, it may be synthesized that:

$$M_x = (R\cdot M_{A1} + R\cdot M_{A2})(t)$$

$$M_y = (R\cdot M_{E1} + R\cdot M_{E2})(t)$$

$$M_z = (M_{R1} + M_{\dot{R}} \cdot t_f)(t)$$

And

$$\sigma_x^2 = (R^2\cdot\sigma_{A1}^2 + R^2\cdot\sigma_{A2}^2 + t_f^2\cdot R^2\cdot\sigma_{\dot{A}}^2)(t)$$

$$\sigma_y^2 = (R^2\cdot\sigma_{E1}^2 + R^2\cdot\sigma_{E2}^2 + t_f^2\cdot R^2\cdot\sigma_{\dot{E}}^2)(t)$$

$$\sigma_z^2 = \left(\sigma_{R1}^2 + t_f^2\cdot\sigma_{\dot{R}}^2 + \left(\frac{R}{V_p}\right)^2\cdot\sigma_V^2\right)(t)$$

Once again, with target vulnerability represented in a plane, these results are applicable only when they can be

projected into that plane. Combined with Eq. (14) and Eq. (15), the distribution parameters become:

$$M_{x'} = \{ R\cdot M_{A1} + R\cdot M_{A2} + \frac{\dot{A}\cdot R}{V_p} (M_{R1} + M_{\dot{R}} \cdot t_f) \}(t)$$

$$M_{y'} = \{ R\cdot M_{E1} + R\cdot M_{E2} + \frac{\dot{E}\cdot R}{V_p} (M_{R1} + M_{\dot{R}} \cdot t_f) \}(t)$$

$$\sigma^2_{x'} = \{ R^2(\sigma^2_{A1} + \sigma^2_{A2} + \sigma^2_{\dot{A}} \cdot t_f^2) + \left(\frac{\dot{A}\cdot R}{V_p}\right)^2 (\sigma^2_{R1} + \sigma^2_{\dot{R}} \cdot t_f^2 + \left(\frac{R}{V_p}\right)^2 \sigma^2_V)\} (t)$$

$$\sigma^2_{y'} = \{ R^2 (\sigma^2_{E1} + \sigma^2_{E2} + \sigma^2_{\dot{E}} \cdot t_f^2) + \left(\frac{\dot{E}\cdot R}{V_p}\right)^2 (\sigma^2_{R1} + \sigma^2_{\dot{R}} \cdot t_f^2 + \left(\frac{R}{V_p}\right)^2 \sigma^2_V) \} (t)$$

and the correlation coefficient is given by:

$$\rho = \frac{1}{\sigma_{x'}\ \sigma_{y'}} \quad \mathrm{Cov}\ (\Delta A', \Delta E').$$

where

$$\mathrm{COV}(\Delta A', \Delta E') = E\ \{ (\Delta A' - E(\Delta A'))(\Delta E' - E(\Delta E')) \}$$

This is easily expanded to show that:

$$\mathrm{COV}(\Delta A', \Delta E') = (\dot{A}\cdot\dot{E})\left(\frac{R}{V_p}\right)^2 \sigma^2_z$$

such that

$$\rho = \frac{1}{\sigma_{x'} \; \sigma_{y'}} (\dot{A} \cdot \dot{E}) \left(\frac{R}{V_p}\right)^2 \sigma_z^2$$

These are the input variables which lead to the solution of Eq. (17) and the value of SSKP. They demonstrate a technique for synthesizing SSKP as a function of its many basic inputs. The task of accurately determining and validating each of these components remains to be accomplished.

It is encouraging to see such a logical and basic approach to the calculation of delivery error variance, but that appeal should not be allowed to cloud the practical aspects of its inclusion or adaptation to the model. The actual variance values used in the quantification of BKP are engineering estimates that may or may not be correct. The question concerning whether it is better to use one estimated variance value for the overall system or a synthesized value accumulated from a number of lesser estimates is no less difficult to answer in this case then for any other modeling problem. The development traced in this appendix appears to have a great deal of merit, but it will be of no real consequence to the computation of BKP unless it is possible to show confidence in the methodology for fixing the values of the delivery error variance components.

LIST OF REFERENCES

1. A Parametric Study of the Advanced Forward Area Air Defense Weapon System (AFAADS), Volume I - Systems Analysis, Data Systems Division Litton Systems, Inc., MS00678, 1970.

2. TACOS II Antiaircraft Gun Sub-Model Formulation, Braddock, Dunn and McDonald, Inc., El Paso, Texas, April 1968.

3. Banash, R. C., Notes on the University of Michigan Analytic Gun Model, Technical Note SY-TN3-70, Systems Analysis Directorate, HQ, U. S. Army Weapons Command, Rock Island, Illinois, April 1970.

4. Banash, R. C., An Analytic Procedure for the Computation of Burst Kill Probabilities in Air Defense, Technical Note SY-TN9-70, Systems Analysis Directorate, HQ, U. S. Army Weapons Command, Rock Island, Illinois, October 1970.

5. Taylor, J. G., Mathematical Models of Combat, Lecture Notes for OA 4654, Naval Postgraduate School, Monterey, California, Chapter 7, February 1975.

6. Progress Report on Forward Area Air Defense, Annex H(C), HQ, U. S. Army Missile Command, Huntsville, Alabama, 1965.

7. Helgert, J. H., "On the Computation of Hit Probability," Operations Research 19, p. 668-684, 1971.

8. Parametric Design/Cost Effectiveness(PDCE) Study on Advanced Forward Area Air Defense Systems (AFAADS) Gun Systems, Final Report, Volume III, Systems Research Laboratory, University of Michigan, August 1969.

9. Small Unit Combat Simulation (DYNTACS X), Air Defense Operations Models, RF 2978 FR 71-2A(U), Air Defense Firing and Lethality Models, by S. H. Parry and J. Rheinfrank III, 1971

10. Hillier, F. S., and Lieberman, G. J., Introduction to Operations Research, p. 66, Holden-Day, Inc., 1972.

11. Meyer, P. L., Introductory Probability and Statistical Applications, 2nd ed., p. 71, Addison-Wesley Publishing Co., 1972.

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